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Relative Asymptotics for Orthogonal Polynomials with Unbounded Recurrence Coefficients

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The asymptotic behavior of the ratio p_n/q_n is given, where $\{p_n(x)\colon n=0, 1, ...\}$ and $\{q_n(x)\colon n=0, 1, 2, ...\}$ are orthogonal polynomials with regularly varying recurrence coefficients that are closely related. The result is applied to some classical polynomials. © 1990 Academic Press, Inc.

1. INTRODUCTION

Let $\{p_n(x)\colon n=0, 1, 2, ...\}$ be a sequence of orthogonal polynomials defined by a recurrence relation

$$xp_n(x)=a_{n+1}\,p_{n+1}(x)+b_n\,p_n(x)+a_n\,p_{n-1}(x) \tag{1.1}$$

and suppose that the recurrence coefficients satisfy

$$\lim_{n\to\infty} a_n/c_n=a>0, \qquad \lim_{n\to\infty} b_n/c_n=b\in\mathbb{R}, \tag{1.2}$$

where c_n is an increasing and positive sequence which is regularly varying with index $\alpha>0$, i.e.,

$$c_n=n^\alpha L(n),$$

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where $L: \mathbb{R}^+ \to \mathbb{R}^+$ is a slowly varying function:

$$\lim_{x \to \infty} L(xt)/L(x) = 1 \qquad \text{for every} \quad t > 0.$$

In [VaGe] the asymptotic behavior of the polynomials $p_n(c_n x)$ is given under the extra condition that

$$\begin{aligned} &\lim_{n \to \infty} n(a_{n+1} - a_n)/c_n = a\alpha \\ &\lim_{n \to \infty} n(b_{n+1} - b_n)/c_n = b\alpha. \end{aligned} \tag{1.3}$$

The result is

$$\lim_{n \to \infty} p_n(c_n x) \Big/ \left(\prod_{k=1}^{n} z_{k,n} \right) = \left\{ \frac{(x-b)^2 - 4a^2}{x^2} \right\}^{1/4}$$
$$\times \exp \left\{ (b/2) \int_0^1 \frac{ds}{\sqrt{(x-bs)^2 - 4a^2 s^2}} \right\}$$

uniformly on compact sets of $\mathbb{C} \setminus [D, E]$, where $[D, E]$ is the convex hull of the union of $\{0\}$ and $[b-2a, b+2a]$, and

$$z_{k,n} = z\left(\frac{c_n x - b_k}{2a_k} \right),$$

with $z(x) = x + \sqrt{x^2 - 1}$. The root is such that z is an analytic function in $\mathbb{C} \setminus [-1, 1]$ for which $|z(x)| > 1$ if $x \in \mathbb{C} \setminus [-1, 1]$. The asymptotic behavior of the product of $z_{k,n}$ can be found explicitly when $a_n = an^\alpha$ and $b_n = bn^\alpha$, but is more complicated if the recurrence coefficients are not so smooth. We will give a method to obtain asymptotics for the ratio

$$p_n(c_n x)/q_n(c_n x),$$

where $\{q_n(x): n = 0, 1, 2, ...\}$ are orthogonal polynomials with recurrence coefficients $\{a_n^0, b_n^0\}$ for which again

$$\lim_{n \to \infty} a_n^0/c_n = a, \qquad \lim_{n \to \infty} b_n^0/c_n = b \tag{1.4}$$

but for which the asymptotic behavior of $q_n(c_n x)$ may be easier to compute (for instance by using the explicit formulas in [VaGe]). Asymptotics of this kind are studied by Nevai [Ne] and Máté–Nevai–Totik [MáNeTo] for converging recurrence coefficients (orthogonal polynomials in $M(a, b)$). It is clear that the recurrence coefficients $\{a_n, b_n\}$ behave asymptotically like the comparison coefficients $\{a_n^0, b_n^0\}$, but in order to make our method work, we need to assume more, namely,

$$\lim_{n\to\infty} n(a_n^0 - a_n)/c_n = A,$$
$$\lim_{n\to\infty} n(b_n^0 - b_n)/c_n = B, \tag{1.5}$$

where A and B are two real numbers. This assumption and the regular variation imply

$$\lim_{n\to\infty} n(a^0_{[ns]} - a_{[ns]})/c_n = As^{\alpha-1}$$
$$\lim_{n\to\infty} n(b^0_{[ns]} - b_{[ns]})/c_n = Bs^{\alpha-1} \tag{1.6}$$

for every $s > 0$, where $[x]$ is the integer part of the real number x. The condition (1.5) is natural: it holds for instance for all Freud weights

$$w(x) = \exp(-x^{2m})$$

with m a positive integer, in which case A and B are zero [MáNeZa]. In this paper we will prove the following result for the polynomials

$$\tilde{p}_n(x) = \prod_{j=1}^{n} (a_j/a_j^0)\, p_n(x).$$

THEOREM. *Suppose* $\{p_n(x)$: $n = 0, 1, 2, ...\}$ *are orthogonal polynomials with recurrence coefficients* $\{a_n$: $n = 1, 2, ...\}$ *and* $\{b_n$: $n = 0, 1, 2, ...\}$ *such that* (1.4) *and* (1.5) *hold, then*

$$\lim_{n\to\infty} \tilde{p}_n(c_n x)/q_n(c_n x) = \exp \int_0^1 \frac{s^{\alpha-1}}{\sqrt{(x-bs^\alpha)^2 - 4a^2 s^{2\alpha}}} \times \left\{ B + \frac{4aAs^\alpha}{x - bs^\alpha + \sqrt{(x-bs^\alpha)^2 - 4a^2 s^{2\alpha}}} \right\} ds \tag{1.7}$$

uniformly on compact sets of $\mathbb{C}\setminus[D, E]$, *where* $[D, E]$ *is the convex hull of* $\{0\}$ *and* $[b-2a, b+2a]$.

The asymptotic behavior on the interval $[D, E]$ is more difficult to handle because the zeros of $p_n(c_n x)$ and of $q_n(c_n x)$ are dense on that interval: one actually knows the asymptotic distribution of those zeros [Va1, pp. 121–124]. This means that both $p_n(c_n x)$ and $q_n(c_n x)$ behave in an oscillatory way on $[D, E]$ and the ratio on the left hand side of (1.7) has many zeros and poles on $[D, E]$. Note that for $b^2-4a^2>0$ the interval $[D, E]$ is different from $[b-2a, b+2a]$. Strong asymptotics for polynomials orthogonal on an infinite interval have recently been obtained. In these studies (see Lubinsky and Saff [LuSa and references therein]) one begins with the weight function instead of the recurrence coefficients.

2. Preliminary Results

Given the sequences $\{a_n^0\}$ and $\{b_n^0\}$, $a_{n+1}^0>0$, $b_n^0\in\mathbb{R}$ $(n=0, 1, 2, \ldots)$ we define $q_n^{(k)}(x)$, $k=0, 1, 2, \ldots$ to be the solution to the following equation

$$a_{n+k+1}^0 q_{n+1}^{(k)}(x)+b_{n+k}^0 q_n^{(k)}(x)+a_{n+k}^0 q_{n-1}^{(k)}(x)=xq_n^{(k)}(x)$$

satisfying the boundary conditions

$$q_{-1}^{(k)}(x)=0, \qquad q_0^{(k)}(x)=1.$$

(note that $q_n^{(0)}(x)=q_n(x)$). One easily verifies that, for k fixed, $q_{m-k}^{(k)}(x)$, $q_{m-k-1}^{(k+1)}(x)$, and $q_{m-k-2}^{(k+2)}(x)$ are three sequences (in the variable m) that satisfy the same recurrence relation of second order. Therefore these three sequences are linearly dependent and hence there exist C_1 and C_2 (possibly depending on k but not on m) such that

$$q_{m-k-1}^{(k+1)}(x)=C_1 q_{m-k}^{(k)}(x)+C_2 q_{m-k-2}^{(k+2)}(x).$$

By choosing $m=k+1$ and $m=k+2$ one obtains C_1 and C_2 and the resulting formula is

$$xq_{m-k-1}^{(k+1)}(x)=\frac{(a_{k+1}^0)^2}{a_{k+2}^0}q_{m-k-2}^{(k+2)}(x)+b_k^0 q_{m-k-1}^{(k+1)}(x)+a_{k+1}^0 q_{m-k}^{(k)}(x). \tag{2.1}$$

Define

$$\tilde{p}_n(x)=\prod_{j=1}^{n}(a_j/a_j^0)\, p_n(x), \tag{2.2}$$

then the recurrence relation (1.1) for these modified polynomials becomes

$$x\tilde{p}_k(x)=a_{k+1}^0\tilde{p}_{k+1}(x)+b_k\tilde{p}_k(x)+\frac{a_k^2}{a_k^0}\tilde{p}_{k-1}(x), \tag{2.3}$$

Multiply (2.1) by $\tilde{p}_k(x)$ and (2.3) by $q_{m-k-1}^{(k+1)}(x)$ and substract the obtained equations, then one finds

$$\tilde{p}_{k+1}(x)\,q_{m-k-1}^{(k+1)}(x)-\tilde{p}_k(x)\,q_{m-k}^{(k)}(x)=\frac{b_k^0-b_k}{a_{k+1}^0}\tilde{p}_k(x)\,q_{m-k-1}^{(k+1)}(x)$$
$$+\frac{a_{k+1}^0}{a_{k+2}^0}\tilde{p}_k(x)\,q_{m-k-2}^{(k+2)}(x)$$
$$-\frac{a_k^2}{a_k^0a_{k+1}^0}\tilde{p}_{k-1}(x)\,q_{m-k-1}^{(k+1)}(x).$$

Summing from $k=0$ to m gives (with the appropriate boundary conditions)

$$\tilde{p}_m(x)=q_m(x)$$
$$+\sum_{k=0}^{m-1}\left\{\frac{b_k^0-b_k}{a_{k+1}^0}q_{m-k-1}^{(k+1)}(x)+\frac{(a_{k+1}^0)^2-(a_{k+1})^2}{a_{k+1}^0a_{k+2}^0}q_{m-k-2}^{(k+2)}(x)\right\}\tilde{p}_k(x). \tag{2.4}$$

This comparison equation will play an essential role in what follows (see also [GeVa, (III.8); Va1, (2.2.7)] for this equation). Changing x into c_nx in (2.4), then dividing by $q_m(c_nx)$, we find

$$\tilde{p}_m(c_nx)/q_m(c_nx)=1+\sum_{k=0}^{m-1}\{B_n(k,m,x)+A_n(k,m,x)\}\,\tilde{p}_k(c_nx)/q_k(c_nx), \tag{2.5}$$

where

$$B_n(k,m,x)=\frac{b_k^0-b_k}{c_n}\,\frac{c_n}{a_{k+1}^0}\,\frac{q_k(c_nx)\,q_{m-k-1}^{(k+1)}(c_nx)}{q_m(c_nx)}, \tag{2.6}$$

$$A_n(k,m,x)=\frac{q_k(c_nx)}{q_{k+1}(c_nx)}\,\frac{a_{k+1}^0-a_{k+1}}{c_n}\,\frac{a_{k+1}^0+a_{k+1}}{a_{k+1}^0}$$
$$\times\frac{c_n}{a_{k+2}^0}\,\frac{q_{k+1}(c_nx)\,q_{m-k-2}^{(k+2)}(c_nx)}{q_m(c_nx)}. \tag{2.7}$$

We can write this by means of an integral as

$$\tilde{p}_{[nt]}(c_nx)/q_{[nt]}(c_nx)=1+\int_0^{[nt]/n}n\{B_n([ns],[nt],x)+A_n([ns],[nt],x)\}$$
$$\times\tilde{p}_{[ns]}(c_nx)/q_{[ns]}(c_nx)\,ds. \tag{2.8}$$

We now need a few lemmas to show that the integrand converges to some function that we can find explicitly.

LEMMA 1 [NeDe, Lemma 3]. *Suppose*

$$\lim_{n\to\infty} a_n^0/c_n = a > 0, \qquad \lim_{n\to\infty} b_n^0/c_n = b,$$

where $c_n = n^\alpha L(n)$, *with* L *slowly varying and* $\alpha > 0$. *If* μ_0 *is a spectral measure for the orthonormal polynomials* $\{q_n(x): n = 0, 1, ...\}$, *then*

$$\lim_{n\to\infty} \int f(x/c_n)\{q_n(x)\}^2\, d\mu_0(x) = \frac{1}{\pi}\int_{b-2a}^{b+2a} \frac{f(x)\,dx}{\sqrt{4a^2-(x-b)^2}} \tag{2.9}$$

for every polynomial f.

COROLLARY 1. *Given the hypothesis of Lemma* 1 *and* $0 \leqslant s < t \leqslant 1$

$$\lim_{n\to\infty} \sum_{j=1}^{[nt]} \lambda_{j,[nt]}\{q_{[ns]}(x_{j,[nt]})\}^2 (x_{j,[nt]}/c_n)^M$$
$$= \frac{1}{\pi}\int_{b-2a}^{b+2a} \frac{(xs^\alpha)^M}{\sqrt{4a^2-(x-b)^2}}\,dx \tag{2.10}$$

for every integer $M \geqslant 0$. *Here*

$$\lambda_{j,n} = \frac{-1}{a_{n+1}^0}\frac{1}{q_{n+1}(x_{j,n})\, q_n'(x_{j,n})} \tag{2.11}$$

and $x_{j,n}$ *are the zeros of* $q_n(x)$.

Proof. Since $t > s$ it follows that for n sufficiently large $2[ns] + M \leqslant 2[nt] - 1$ so that by the Gauss–Jacobi quadrature formula

$$\sum_{j=1}^{[nt]} \lambda_{j,[nt]}\{q_{[ns]}(x_{j,[nt]})\}^2 (x_{j,[nt]}/c_n)^M$$
$$= \left(\frac{1}{c_n}\right)^M \int \{q_{[ns]}(x)\}^2\, x^M\, d\mu_0(x).$$

The result now follows using Lemma 1 and the fact that

$$\lim_{n\to\infty} \frac{c_{[ns]}}{c_n} = s^\alpha.$$

LEMMA 2. *Suppose*

$$\lim_{n\to\infty} a_n^0/c_n = a > 0, \qquad \lim_{n\to\infty} b_n^0/c_n = b,$$

where $c_n = n^\alpha L(n)$, *with* L *slowly varying and* $\alpha > 0$. *Then for* $0 < s \leqslant 1$,

$$\lim q_{[ns]}(c_n x)/q_{[ns]+1}(c_n x) = \frac{2as^\alpha}{x - bs^\alpha + \sqrt{(x-bs^\alpha)^2 - 4a^2 s^{2\alpha}}} \tag{2.12}$$

uniformly on compact sets of $\mathbb{C}\setminus[D, E]$, *where* $[D, E]$ *is the convex hull of* $\{0\}$ *and* $[b-2a, b+2a]$.

Proof. See [Va1, p. 117].

LEMMA 3. *Suppose*

$$\lim_{n\to\infty} a_n^0/c_n = a > 0, \qquad \lim_{n\to\infty} b_n^0/c_n = b,$$

where $c_n = n^\alpha L(n)$, *with* L *slowly varying and* $\alpha > 0$. *Then for* $1 \geqslant t > s \geqslant 0$

$$\frac{c_n}{a^0_{[ns]+1}} \frac{q_{[ns]}(c_n x)\, q^{([ns]+1)}_{[nt]-[ns]-1}(c_n x)}{q_{[nt]}(c_n x)} \to \frac{1}{\sqrt{(x-bs^\alpha)^2 - 4a^2 s^{2\alpha}}} \tag{2.13}$$

uniformly on compact sets of $\mathbb{C}\setminus[D, E]$ *where* $[D, E]$ *is the convex hull of* $\{0\}$ *and* $[b-2a, b+2a]$.

Proof. Let us first of all note that the function appearing on the left of (2.13) is a rational function with poles at the zeros of the denominator. A decomposition into partial fractions gives

$$\frac{1}{a^0_{k+1}} \frac{q_k(x)\, q^{(k+1)}_{n-k-1}(x)}{q_n(x)} = \sum_{j=1}^{n} \frac{b_{j,n}}{x - x_{j,n}}, \tag{2.14}$$

where

$$b_{j,n} = \frac{1}{a^0_{k+1}} \frac{q_k(x_{j,n})\, q^{(k+1)}_{n-k-1}(x_{j,n})}{q_n'(x_{j,n})}$$

and $x_{j,n}$ are the zeros of $q_n(x)$. Since $q_n(x)$ and $q^{(k+1)}_{n-k-1}(x)$ satisfy the same recurrence formula for $n \geqslant k$, the following Wronskian identity holds

$$a^0_{n+1}\{q_n(x)\, q^{(k+1)}_{n-k}(x) - q_{n+1}(x)\, q^{(k+1)}_{n-k-1}(x)\} = a^0_{k+1} q_k(x)$$

so that the residues are given by

$$b_{j,n} = \lambda_{j,n}\{q_k(x_{j,n})\}^2, \tag{2.15}$$

where $\lambda_{j,n}$ is given by (2.11). Therefore the left hand side of (2.13) becomes

$$\sum_{j=1}^{[nt]} \frac{\lambda_{j,[nt]}\{q_{[ns]}(x_{j,[nt]})\}^2}{x - x_{j,[nt]}/c_n} \tag{2.16}$$

Since

$$|x_{j,n}| \leqslant \max_{0 \leqslant j \leqslant n-1} |b_j| + 2 \max_{0 \leqslant j \leqslant n-1} a_{j+1} \tag{2.17}$$

(see [NeDe, p. 1188]) (2.16) can be rewritten for $|x|$ sufficiently large as

$$= \sum_{k=0}^{m} \sum_{j=1}^{[nt]} \lambda_{j,[nt]}\{q_{[ns]}(x_{j,[nt]})\}^2 \frac{(x_{j,[nt]}/c_n)^k}{x^{k+1}}$$
$$+ \sum_{j=1}^{[nt]} \frac{\lambda_{j,[nt]}\{q_{[ns]}(x_{j,[nt]})\}^2}{x - x_{j,[nt]}/c_n} \left(\frac{x_{j,[nt]}/c_n}{x}\right)^{m+1}$$

Consequently, for every m and $|x|$ sufficiently large (2.17) and Corollary 1 imply that

$$\lim_{n\to\infty} \left| \sum_{j=1}^{[nt]} \frac{\lambda_{j,[nt]} q_{[ns]}(x_{j,[nt]})^2}{x - x_{j,[nt]}/c_n} - \sum_{k=0}^{m} \frac{1}{x^{k+1}} \frac{1}{\pi} \int_{b-2a}^{b+2a} \frac{(ys^\alpha)^k \, dy}{\sqrt{4a^2 - (y-b)^2}} \right|$$
$$\leqslant \frac{(\gamma s^\alpha)^{m+1}}{|x|^{m+1} d}.$$

Here $\gamma = |b| + 2a$ and d is the distance from x to $[D, E]$. Now letting $m \to \infty$ and using the fact that

$$\frac{1}{\pi} \int_{b-2a}^{b+2a} \frac{1}{\sqrt{4a^2 - (y-b)^2}} \frac{dy}{x - ys^\alpha} = \frac{1}{\sqrt{(x - bs^\alpha)^2 - 4a^2 s^{2\alpha}}}$$

gives the result for $|x|$ sufficiently large. Now

$$\left| \sum_{j=1}^{[nt]} \frac{\lambda_{j,[nt]}\{q_{[ns]}(x_{j,[nt]})\}^2}{x - x_{j,[nt]}/c_n} \right| \leqslant \frac{1}{d}$$

for n sufficiently large. This coupled with the theorem of Stieltjes–Vitali (see [Ch, p. 121]) extends the result to $\mathbb{C} \setminus [D, E]$.

LEMMA 4. *Suppose* $\{p_n(x)\colon n = 0, 1, 2, ...\}$ *are orthogonal polynomials with recurrence coefficients* $\{a_n\colon n = 1, 2, ...\}$ *and* $\{b_n\colon n = 0, 1, 2, ...\}$ *such that* (1.4) *and* (1.5) *hold, then for every* $m \leqslant n$ *and for* x *in a compact set* K

of $\mathbb{C}\setminus[D, E]$, *with* $[D, E]$ *the convex hull of* $\{0\}$ *and* $[b-2a, b+2a]$, *one has*

$$\left|\frac{\tilde{p}_m(c_n x)}{q_m(c_n x)}\right| \leqslant \exp M(n, m), \tag{2.18}$$

where

$$M(n, m) = \frac{M}{\delta c_n} \sum_{k=0}^{m-1} \frac{c_k}{(k+1)} + \frac{M^2}{(\delta c_n)^2} \sum_{k=0}^{m-1} \frac{(c_k)^2}{(k+1)}$$

with M *a positive constant and* δ *the distance between* K *and* $[D, E]$. *If* $j \leqslant m \leqslant n$, *then*

$$\left|\frac{\tilde{p}_m(c_n x)}{q_m(c_n x)} - \frac{\tilde{p}_j(c_n x)}{q_j(c_n x)}\right| \leqslant \{M(n, m) - M(n, j)\} \exp\{M(n, m)\}$$

$$+ \exp\{M(n, j)\} \frac{M'}{\delta^3 c_n}$$

$$\times \max_{0 \leqslant k < j} \frac{c_k}{(k+1)} \left\{1 + \left|\frac{q_{j+1}(c_n x)}{q_j(c_n x)}\right|\right\}, \tag{2.19}$$

with M' *some positive constant, and*

$$\left|\frac{\tilde{p}_m(c_n x)}{q_m(x_n x)} - 1\right| \leqslant M(n, m) \exp\{M(n, m)\}. \tag{2.20}$$

Proof. In order to prove (2.18) we will use induction on m. For $m=0$ both the left hand side and the right hand side are equal to one. Suppose next that (2.18) is true for all integers up to $m-1$, then from (2.5)

$$\left|\frac{\tilde{p}_m(c_n x)}{q_m(c_n x)}\right| \leqslant 1 + \sum_{k=0}^{m-1} \{|B_n(k, m, x)| + |A_n(k, m, x)|\} \exp\{M(n, k)\}. \tag{2.21}$$

Let M be such that for every k

$$(k+1)\left|\frac{b_k^0 - b_k}{c_k}\right| \leqslant M, \quad (k+1)\left|\frac{a_{k+1}^0 - a_{k+1}}{c_k}\right| \leqslant M, \quad \left|\frac{a_{k+1}^0 + a_{k+1}}{c_k}\right| \leqslant M, \tag{2.22}$$

and let δ be the distance between the compact set K and the convex hull

$[D, E]$ of $\{0\}$ and $[b-2a, b+2\text{a}]$, then $|x - x_{j,m}/c_n| > \delta$ for $x \in K$ and $m \leqslant n$. Now (2.14), (2.15) and the fact that

$$\sum_{j=1}^{m} \lambda_{j,m} q_k(x_{j,m})^2 = 1, \qquad 0 \leqslant k \leqslant m-1$$

imply that

$$|B_n(k, m, x)| \leqslant \left| \frac{b_k^0 - b_k}{c_n} \right| \sum_{j=1}^{m} \frac{\lambda_{j,m} \{q_k(x_{j,m})\}^2}{|x - x_{j,m}/c_n|} \leqslant \frac{M}{\delta c_n} \frac{c_k}{(k+1)} \tag{2.23}$$

Equation (2.22) has been used to obtain the last inequality. In a similar way

$$\begin{aligned} |A_n(k, m, x)| \leqslant & \left| \frac{a_{k+1}^0 - a_{k+1}}{c_n} \right| \frac{a_{k+1}^0 + a_{k+1}}{c_n} \\ & \times \sum_{j=1}^{k+1} \frac{\lambda_{j,k+1} \{q_k(x_{j,k+1})\}^2}{|x - x_{j,k+1}/c_n|} \sum_{j=1}^{m} \frac{\lambda_{j,m} \{q_{k+1}(x_{j,m})\}^2}{|x - x_{j,m}/c_n|} \\ \leqslant & \frac{M^2}{(\delta c_n)^2} \frac{(c_k)^2}{(k+1)}. \end{aligned} \tag{2.24}$$

Using (2.23) and (2.24) in (2.11) gives

$$\left| \frac{\tilde{p}_m(c_n x)}{q_m(c_n x)} \right| \leqslant 1 + \sum_{k=0}^{m-1} \left\{ \frac{M}{\delta c_n} \frac{c_k}{(k+1)} + \frac{M^2}{(\delta c_n)^2} \frac{(c_k)^2}{(k+1)} \right\} \exp\{M(n, k)\}.$$

Then, using the inequality $x < e^x - 1$ with

$$x = \frac{M}{\delta c_n} \frac{c_k}{(k+1)} + \frac{M^2}{(\delta c_n)^2} \frac{(c_k)^2}{(k+1)},$$

we find

$$\left| \frac{\tilde{p}_m(c_n x)}{q_m(c_n x)} \right| \leqslant 1 + \sum_{k=0}^{m-1} [\exp\{M(n, k+1)\} - \exp\{M(n, k)\}].$$

This sum is telescoping and results in the inequality (2.18). In order to prove (2.19) we use (2.5) to find

$$\begin{aligned} \left| \frac{\tilde{p}_m(c_n x)}{q_m(c_n x)} - \frac{\tilde{p}_j(c_n x)}{q_j(c_n x)} \right| \leqslant & \sum_{k=j}^{m-1} \{|B_n(k, m, x)| + |A_n(k, m, x)|\} \left| \frac{\tilde{p}_k(c_n x)}{q_k(c_n x)} \right| \\ & + \sum_{k=0}^{j-1} \{|B_n(k, m, x) - B_n(k, j, x)| \\ & + |A_n(k, m, x) - A_n(k, j, x)|\} \left| \frac{\tilde{p}_k(c_n x)}{q_k(c_n x)} \right|. \end{aligned} \tag{2.25}$$

The first sum on the right hand side of (2.25) can easily be bounded using (2.18), (2.23), (2.24) and the fact that $M(n, k) \leqslant M(n, m)$ for every $k \leqslant m$, giving

$$\sum_{k=j}^{m-1} \{|B_n(k, m, x)| + |A_n(k, m, x)|\} \left|\frac{\tilde{p}_k(c_n x)}{q_k(c_n x)}\right|$$
$$\leqslant \exp\{M(n, m)\} \left\{\frac{M}{\delta c_n} \sum_{k=j}^{m-1} \frac{c_k}{k+1} + \frac{M^2}{(\delta c_n)^2} \sum_{k=j}^{m-1} \frac{(c_k)^2}{k+1}\right\}.$$

By using (2.6) and (2.7) we also find

$$B_n(k, m, x) - B_n(k, j, x) = \frac{b_k^0 - b_k}{a_{k+1}^0}\, q_k \left\{\frac{q_{m-k-1}^{(k+1)}}{q_m} - \frac{q_{j-k-1}^{(k+1)}}{q_j}\right\}$$

and

$$A_n(k, m, x) - A_n(k, j, x) = \frac{a_{k+1}^0 - a_{k+1}}{a_{k+2}^0}\, \frac{a_{k+1}^0 + a_{k+1}}{a_{k+1}^0}$$
$$\times\, q_k \left\{\frac{q_{m-k-2}^{(k+2)}}{q_m} - \frac{q_{j-k-2}^{(k+2)}}{q_j}\right\},$$

where we have dropped the argument $c_n x$ for convenience. In order to simplify these last two expressions, we will use the formula

$$q_{m-k-1}^{(k+1)} q_j - q_{j-k-1}^{(k+1)} q_m = \frac{a_{k+1}^0}{a_{j+1}^0}\, q_k\, q_{m-j-1}^{(j+1)}. \tag{2.26}$$

This formula is true because for j and k fixed both sides of (2.26) satisfy the same recurrence relation and for $m = j$ both sides of the equation are zero, while for $m = j + 1$ the left hand side is

$$q_{j-k}^{(k+1)} q_j - q_{j-k-1}^{(k+1)} q_{j+1}.$$

This expression multiplied by a_{j+1}^0 is exactly the Wronskian of the two solutions $\{q_j\colon j = 0, 1, ...\}$ and $\{q_{j-k-1}^{(k+1)}\colon j = 0, 1, ...\}$ and since the Wronskian is independent of j, one may choose $j = k$ to evaluate the expression, giving

$$q_{j-k}^{(k+1)} q_j - q_{j-k-1}^{(k+1)} q_{j+1} = \frac{a_{k+1}^0}{a_{j+1}^0}\, q_k$$

which is exactly the right hand side of (2.26) for $m=j+1$. Using (2.26) then gives

$$B_n(k,m,x)-B_n(k,j,x)=\frac{b_k^0-b_k}{a_{j+1}^0}\left\{\frac{q_k}{q_j}\right\}^2\frac{c_n}{a_{j+1}^0}\frac{q_jq_{m-j-1}^{(j+1)}}{q_m}$$

$$A_n(k,m,x)-A_n(k,j,x)=\frac{a_{k+1}^0-a_{k+1}}{c_n^0}\frac{a_{k+1}^0+a_{k+1}}{a_{k+1}^0}$$

$$\times\frac{q_kq_{k+1}}{\{q_j\}^2}\frac{c_n}{a_{j+1}^0}\frac{q_jq_{m-j-1}^{(j+1)}}{q_m}.$$

Straightforward estimates lead to

$$|B_n(k,m,x)-B_n(k,j,x)|\leqslant\frac{M}{\delta c_n}\frac{c_k}{k+1}\left|\frac{q_k}{q_j}\right|^2,$$

$$|A_n(k,m,x)-A_n(k,j,x)|\leqslant\frac{M^2}{\delta c_n}\frac{c_k}{k+1}\left|\frac{q_kq_{k+1}}{\{q_j\}^2}\right|.$$

This means that the second sum on the right hand side of (2.25) can be bounded by

$$\sum_{k=0}^{j-1}\{|B_n(k,m,x)-B_n(k,j,x)|+|A_n(k,m,x)-A_n(k,j,x)|\}\left|\frac{\tilde{p}_k(c_nx)}{q_k(c_nx)}\right|$$

$$\leqslant\exp\{M(n,j)\}\frac{M}{\delta c_n|q_j|^2}\max_{0\leqslant k\leqslant j}\frac{c_k}{k+1}\sum_{k=0}^{j-1}\{|q_k|^2+|q_kq_{k+1}|\}.$$

Now use the expression

$$\frac{\sum_{k=0}^{j-1}|q_k(c_nx)|^2}{|q_j(c_nx)|^2}=\left(\frac{a_j^0}{c_n}\right)^2\sum_{k=1}^{j}\lambda_{k,j}\frac{\{q_{j-1}(x_{k,j})\}^2}{|x-x_{k,j}/c_n|^2}$$

[Ne, p. 28] to obtain the bounds

$$\sum_{k=0}^{j-1}|q_k(c_nx)|^2\leqslant\frac{(a_j^0)^2}{(\delta c_n)^2}|q_j(c_nx)|^2$$

and

$$\sum_{k=0}^{j-1}|q_k(c_nx)\,q_{k+1}(c_nx)|\leqslant\left\{\sum_{k=0}^{j-1}|q_k(c_nx)|^2\sum_{k=0}^{j-1}|q_{k+1}(c_nx)|^2\right\}^{1/2}$$

$$\leqslant\frac{a_j^0a_{j+1}^0}{(\delta c_n)^2}|q_j(c_nx)\,q_{j+1}(c_nx)|.$$

Using these bounds gives

$$\sum_{k=0}^{j-1} \{|B_n(k, m, x) - B_n(k, j, x)| + |A_n(k, m, x) - A_n(k, j, x)|\} \left|\frac{\tilde{p}_k(c_n x)}{q_k(c_n x)}\right|$$
$$\leqslant \exp\{M(n, j)\} \frac{M}{\delta c_n} \max_{0 \leqslant k \leqslant j} \frac{c_k}{k+1} \left\{\frac{(a_j^0)^2}{(\delta c_n)^2} + \frac{a_j^0 a_{j+1}^0}{(\delta c_n)^2} \left|\frac{q_{j+1}(c_n x)}{q_j(c_n x)}\right|\right\},$$

which then gives the desired inequality (2.19). Finally, we turn to the proof of (2.20). From (2.5) and (2.18) we find

$$\left|\frac{\tilde{p}_m(c_n x)}{q_m(c_n x)} - 1\right| \leqslant \sum_{k=0}^{m-1} \{|B_n(k, m, x)| + |A_n(k, m, x)|\} \exp\{M(n, k)\}. \tag{2.27}$$

Now $M(n, k) \leqslant M(n, m)$ for every $k \leqslant m$, and by using (2.23) and (2.24) the inequality (2.20) follows.

3. Proof of the Theorem

We will first show that the limit exists. Denote by $B([0, 1])$ the space of bounded Borel measurable functions on $[0, 1]$ with the supremum norm. Define the sequence $\{f_n(t): n = 1, 2, ...\}$ in $B([0, 1])$ by

$$f_n(t) = \frac{\tilde{p}_{[nt]}(c_n x)}{q_{[nt]}(c_n x)}, \tag{3.1}$$

with x in a compact set K of $\mathbb{C} \setminus [D, E]$ where $[D, E]$ is the convex hull of $\{0\}$ and $[b - 2a, b + 2a]$. In what follows all results hold uniformly for $x \in K$. Standard properties of regularly varying sequences [BoSe, BiGoTe, Se] say that for a regularly varying sequence $\{d_n: n = 1, 2, ...\}$ with index $\beta > 0$ one has

$$\lim_{n \to \infty} \frac{1}{d_n} \sum_{k=0}^{n} \frac{d_k}{k+1} = \frac{1}{\beta}. \tag{3.2}$$

and

$$\lim_{n \to \infty} \frac{1}{d_n} \max_{0 \leqslant k \leqslant n} \frac{d_k}{k} = 0. \tag{3.3}$$

This and (2.18) show that the sequence $\{f_n\}$ is bounded. The inequalities (2.19) and (2.20) and the properties (3.2) and (3.3) also show that for every $\varepsilon > 0$ there is a finite collection $\{E_1, ..., E_m\}$ of disjoint sets in $[0, 1]$ with union $[0, 1]$ and points s_i in E_i such that for every integer n and for $i=1, \ldots, m$

$$\sup_{s \in E_i} |f_n(s) - f_n(s_i)| \leqslant \varepsilon.$$

This means that the sequence $\{f_n\}$ is sequentially compact [DuSc, p. 260] and hence there exists a subsequence n_i such that

$$f(t) = \lim_{i \to \infty} \frac{\tilde{p}_{[n_i t]}(c_{n_i}, x)}{q_{[n_i t]}(c_{n_i}, x)}$$

exists, and from Lemma 4 one finds that this limit is continuous on $[0, 1]$. We will show that every converging subsequence has the same limit so that the sequence $\{f_n\}$ converges to this limit f. By (1.5) and Lemma 3 we find that for $1 \geqslant t > s \geqslant 0$

$$\lim_{n \to \infty} nB_n([ns], [nt], x) = \frac{Bs^{\alpha-1}}{\sqrt{(x - bs^\alpha)^2 - 4a^2 s^{2\alpha}}} = B(s, x)$$

and if we use Lemma 3 with s replaced by $s + 1/n$ and (1.5) then

$$\begin{aligned} &\lim_{n \to \infty} nA_n([ns], [nt], x) \\ &= \frac{4Aas^{2\alpha-1}}{\sqrt{(x - bs^\alpha)^2 - 4a^2 s^{2\alpha}}\,\{x - bs^\alpha + \sqrt{(x - bs^\alpha)^2 - 4a^2 s^{2\alpha}}\}} \\ &= A(s, x). \end{aligned}$$

Hence if we take limits in (2.8) (Lebesgue's theorem can be used because of Lemma 4 and the bounds (2.23) and (2.24)) we will find that for $0 < t \leqslant 1$

$$f(t) = 1 + \int_0^t \{B(s, x) + A(s, x)\}\, f(s)\, ds. \tag{3.4}$$

Clearly $f(t)$ is differentiable in t, so that differentiation with respect to t gives

$$f'(t) = \{B(t, x) + A(t, x)\}\, f(t)$$

and the unique solution of this differential equation with $f(0) = 1$ is given by

$$f(t) = \exp \int_0^t \{B(s, x) + A(s, x)\}\, ds.$$

Hence all possible limits are the same. Of particular interest is the quantity $f(1)$ which is exactly the expression on the right hand side of (1.7).

The proof of this theorem simplifies substantially if one assumes that the recurrence coefficients are "smooth" in the sense that (1.3) holds. In that case the polynomials $\{p_n\}$ and $\{q_n\}$ have asymptotic behavior given by [VaGe, Thm 2]

$$\lim_{n \to \infty} \frac{p_n(c_n x)}{(\prod_{k=1}^n z_{k,n})} = \lim_{n \to \infty} \frac{q_n(c_n x)}{(\prod_{k=1}^n z_{k,n}^0)},$$

where

$$z_{k,n} = z\left(\frac{c_n x - b_k}{2a_k}\right), \qquad z_{k,n}^0 = z\left(\frac{c_n x - b_k^0}{2a_k^0}\right).$$

Therefore it is sufficient to investigate the asymptotic behavior of the ratio

$$\frac{(\prod_{k=1}^n z_{k,n})}{(\prod_{k=1}^n z_{k,n}^0)}$$

which can easily be done without using Lemmas 1–4.

4. Some Examples

Example 1: Laguerre Polynomials. The recurrence coefficients for Laguerre polynomials $L_n^{(\alpha)}(x)$ are given by

$$a_n = \sqrt{n(n+\alpha)}$$
$$b_n = 2n + \alpha + 1,$$

where $\alpha > -1$ (do not confuse this α with the index of regular variation for the sequence c_n). Denote the normalized Laguerre polynomials by

$$p_n(x) = (-1)^n \sqrt{n!/(\alpha+1)_n}\, L_n^{(\alpha)}(x).$$

As a comparison system we will use the orthogonal polynomials $q_n(x)$ with recurrence coefficients

$$a_n^0 = n$$
$$b_n^0 = 2n.$$

Clearly $c_n = n$, $a = 1$, $b = 2$, $A = -\alpha/2$, and $B = -\alpha - 1$ for this particular case. Note that $b^2 - 4a^2 = 0$. For the comparison system we have [VaGe]

$$\lim_{n \to \infty} \sqrt{2\pi n}\, \frac{q_n(nx)}{(zH)^n} = \sqrt{z}\, \{x^2 - 4x\}^{-1/4} \exp\left\{\int_0^1 \frac{ds}{\sqrt{x^2 - 4xs}}\right\} \tag{4.1}$$

uniformly on compact sets of $\mathbb{C} \setminus [0, 4]$, where

$$z = \tfrac{1}{2}(x - 2 + \sqrt{x^2 - 4x})$$

and

$$H(x) = \exp\left\{x \int_0^1 \frac{ds}{\sqrt{x^2 - 4xs}}\right\}.$$

By the theorem we have

$$\lim_{n \to \infty} \frac{\tilde{p}_n(nx)}{q_n(nx)} = \exp\left\{-\int_0^1 \frac{1}{\sqrt{x^2 - 4xs}} \left(\alpha + 1 + \frac{2\alpha s}{x - 2s + \sqrt{x^2 - 4xs}}\right) ds\right\},$$

where

$$\tilde{p}_n(x) = \sqrt{(\alpha + 1)_n / n!}\; p_n(x) = (-1)^n L_n^{(\alpha)}(x).$$

Some straightforward integral calculus gives

$$\int_0^1 \frac{ds}{\sqrt{x^2 - 4xs}} = \frac{2}{x + \sqrt{x^2 - 4x}}.$$

Moreover

$$\begin{aligned}
&\int_0^1 \frac{1}{\sqrt{x^2 - 4xs}} \left\{\alpha + 1 + \frac{2\alpha s}{x - 2s + \sqrt{x^2 - 4xs}}\right\} ds \\
&= \int_0^1 \frac{1}{\sqrt{x^2 - 4xs}} \left\{\alpha + 1 + \frac{\alpha(x - 2s - \sqrt{x^2 - 4xs})}{2s}\right\} ds \\
&= \int_0^1 \frac{1}{\sqrt{x^2 - 4xs}}\, ds - \alpha \int_0^1 \frac{d(x + \sqrt{x^2 - 4xs})}{x + \sqrt{x^2 - 4xs}}
\end{aligned}$$

Therefore

$$\lim_{n \to \infty} \frac{\tilde{p}_n(nx)}{q_n(nx)} = \exp\left\{\frac{-2}{x + \sqrt{x^2 - 4x}}\right\} \{x + \sqrt{x^2 - 4x}\}^{\alpha} (2x)^{-\alpha}$$

In combination with (4.1) this gives

$$\lim_{n \to \infty} 2^n \sqrt{2\pi n}\,(-1)^n L_n^{(\alpha)}(nx)\{x-2+\sqrt{x^2-4x}\}^{-n} \exp\left\{\frac{-2nx}{x+\sqrt{x^2-4x}}\right\}$$
$$= 2^{-\alpha-1/2} x^{-\alpha}\{x-2+\sqrt{x^2-4x}\}^{1/2}$$
$$\times \{x+\sqrt{x^2-4x}\}^{\alpha}\{x^2-4x\}^{-1/4}. \tag{4.2}$$

This is in agreement with the asymptotic formula obtained in [MaVa] (see also [Va1, p. 92]) if one takes into account that the square root in the above formula is negative if x is negative. The obtained asymptotic formula also agrees with the Plancherel–Rotach type formula given in Szegö [Sz, p. 175].

EXAMPLE 2: Meixner Polynomials. The recurrence coefficients for Meixner polynomials (Meixner polynomials of the first kind, in Chihara's terminology [Ch]) $m_n(x; \beta, c)$ are

$$a_n = \frac{\sqrt{c}}{1-c}\sqrt{n(n+\beta-1)}$$
$$b_n = \frac{(1+c)n+\beta c}{1-c},$$

where $0 < c < 1$ and $\beta > 0$. The normalized Meixner polynomials are

$$p_n(x) = (-1)^n \frac{c^{n/2}}{\sqrt{n!\,(\beta)_n}} m_n(x; \beta, c).$$

As a comparison system we now use the orthogonal polynomials $q_n(x)$ with recurrence coefficients

$$a_n^0 = \frac{\sqrt{c}}{1-c} n$$
$$b_n^0 = \frac{1+c}{1-c} n$$

so that $c_n = n$,

$$a = \frac{\sqrt{c}}{1-c}, \qquad b = \frac{1+c}{1-c},$$
$$A = \frac{-\sqrt{c}}{1-c}\frac{\beta-1}{2}, \qquad B = -\frac{\beta c}{1-c}.$$

Note that $b^2 - 4a^2 = 1$. The asymptotic behavior of the comparison system is given by

$$\lim_{n\to\infty} \sqrt{2\pi n}\, \frac{q_n(nx)}{(zH)^n} = (az)^{1/2} \{x^2 - 2bx + 1\}^{-1/4} \times \exp\left\{\frac{b}{2}\int_0^1 \frac{ds}{\sqrt{x^2 - 2bx + s^2}}\right\}$$

uniformly on compact sets of $\mathbb{C}\setminus[0, b+2a]$, where

$$z = \frac{\{x - b + \sqrt{x^2 - 2bx + 1}\}}{2a},$$

$$H = \exp\left\{x \int_0^1 \frac{ds}{\sqrt{x^2 - 2bxs + s^2}}\right\}.$$

The relative asymptotic behavior is given by

$$\lim_{n\to\infty} \frac{\tilde{p}_n(nx)}{q_n(nx)} = \exp\left\{-\int_0^1 \frac{1}{\sqrt{x^2 - 2bsx + s^2}} \times \left(\frac{\beta c}{1-c} + \frac{2c}{(1-c)^2} \frac{(\beta - 1)s}{x - bs + \sqrt{x^2 - 2bsx + s^2}}\right) ds\right\},$$

where $\tilde{p}_n(x) = \sqrt{(\beta)_n/n!}\, p_n(x) = (-1)^n c^{n/2}/n!\, m_n(x; \beta, c)$.
In order to get rid of the integrals appearing in these formulas, we use (with a and b given above)

$$\int_0^1 \frac{ds}{\sqrt{x^2 - 2bsx + s^2}} = \int_0^1 \frac{\frac{d}{ds}(-bx + s + \sqrt{x^2 - 2bsx + s^2})}{-bx + s + \sqrt{x^2 - 2bsx + s^2}}\, ds$$
$$= \log\left\{\frac{-bx + 1 + \sqrt{x^2 - 2bx + 1}}{(1-b)x}\right\}. \tag{4.3}$$

The expression appearing in the relative asymptotic becomes

$$-\int_0^1 \frac{1}{\sqrt{x^2 - 2bsx + s^2}} \left(\frac{\beta c}{1-c} + \frac{2c}{(1-c)^2} \frac{(\beta-1)s}{x - bs + \sqrt{x^2 - 2bsx + s^2}}\right) ds$$
$$= -\int_0^1 \frac{1}{\sqrt{x^2 - 2bsx + s^2}} \left(\frac{\beta c}{1-c} + \frac{2c(\beta-1)s}{(1-c)^2} \frac{x - bs - \sqrt{x^2 - 2bxs + s^2}}{4a^2 s^2}\right) ds$$
$$= -\int_0^1 \frac{1}{\sqrt{x^2 - 2bsx + s^2}} \left(\frac{b-1}{2} + \frac{\beta - 1}{2} \frac{x - s - \sqrt{x^2 - 2bxs + s^2}}{s}\right) ds.$$

These integrals can then be calculated by (4.3) and

$$\int_0^1 \frac{x-s-\sqrt{x^2-2bxs+s^2}}{\sqrt{x^2-2bxs+s^2}}\,\frac{ds}{s}$$

$$= \int_0^1 \frac{x-s-\sqrt{x^2-2bxs+s^2}}{s}\,\frac{\frac{d}{ds}\left(x+s-\sqrt{x^2-2bxs+s^2}\right)}{-bx+s+\sqrt{x^2-2bxs+s^2}}\,ds$$

$$= 2x(b-1)$$

$$\times \int_0^1 \frac{\frac{d}{ds}\left(x+s-\sqrt{x^2-2bxs+s^2}\right)}{\left(x-s-\sqrt{x^2-2bxs+s^2}\right)\left(-bx+s+\sqrt{x^2-2bxs+s^2}\right)}\,ds$$

$$= -2\int_0^1 \frac{\frac{d}{ds}\left(x+s+\sqrt{x^2-2bxs+s^2}\right)}{x+s+\sqrt{x^2-2bxs+s^2}}\,ds$$

$$= -2\log\left\{\frac{x+1+\sqrt{x^2-2bx+1}}{2x}\right\}.$$

All this eventually results in the asymptotic formula

$$\lim_{n\to\infty}\,(-1)^n\sqrt{2\pi n}\,c^{n/2}\,\frac{m_n(nx;\beta,c)}{n!}\,(2a)^n\left\{x-b+\sqrt{x^2-2bx+1}\right\}^{-n}$$

$$\times\left\{\frac{-bx+1+\sqrt{x^2-2bx+1}}{x(1-b)}\right\}^{-nx}$$

$$=\left\{\frac{x-b+\sqrt{x^2-2bx+1}}{2}\right\}^{1/2}\left\{x^2-2bx+1\right\}^{-1/4}$$

$$\times\left\{\frac{-bx+1+\sqrt{x^2-2bx+1}}{x(1-b)}\right\}^{1/2}$$

$$\times\left\{\frac{x+1+\sqrt{x^2-2bx+1}}{2x}\right\}^{\beta-1}, \tag{4.4}$$

uniformly on compact sets of $\mathbb{C}\backslash[0, b+2a]$. This was already obtained for $x<0$ in [MaVa; Va1, p. 97]. If one wants to check that the formula obtained there and (4.4) are the same, then one needs to take into account that the square root in (4.4) is negative for x negative, so that the function Φ in [MaVa, and Va1] in our notation is

$$\Phi(x) = -(1-c)\sqrt{x^2-2bx+1}.$$

EXAMPLE 3: Meixner–Pollaczek Polynomials. The recurrence coefficients for Meixner–Pollaczek polynomials (Meixner polynomials of the second kind [Ch]) $M_n(x; \delta, \eta)$ are

$$a_n = \sqrt{\delta^2+1}\,\sqrt{n(n+\eta-1)},$$
$$b_n = (2n+\eta)\,\delta,$$

where $\delta \in \mathbb{R}$ and $\eta > 0$. The normalized Meixner–Pollaczek polynomials are

$$p_n(x) = \frac{(\delta^2+1)^{n/2}}{\sqrt{n!\,(\eta)_n}}\, M_n(x; \delta, \eta).$$

We can use the comparison system

$$a_n^0 = \sqrt{\delta^2+1}\; n$$
$$b_n^0 = 2n$$

so that $c_n = n$ and

$$a = \sqrt{\delta^2+1}, \qquad b = 2\delta,$$
$$A = -\frac{\sqrt{\delta^2+1}\,(\eta-1)}{2}, \qquad B = -\eta\delta.$$

Note that $b^2 - 4a^2 = -4$. The asymptotic behavior of the comparison system is given by

$$\lim_{n\to\infty} \sqrt{2\pi n}\,\frac{q_n(nx)}{(zH)^n} = (az)^{1/2}\,\{x^2-4\delta x-4\}^{1/4} \times \exp\left\{\frac{b}{2}\int_0^1 \frac{ds}{\sqrt{x^2-4\delta xs-4s^2}}\right\}$$

uniformly on compact sets of $\mathbb{C}\setminus[b-2a, b+2a]$, where

$$z = \frac{\{x-2\delta+\sqrt{x^2-4\delta x-4}\}}{2a},$$
$$H = \exp\left\{x\int_0^1 \frac{ds}{\sqrt{x^2-2\delta xs-4s^2}}\right\}.$$

The relative asymptotic behavior is given by

$$\lim_{n\to\infty} \frac{\tilde{p}_n(nx)}{q_n(nx)} = \exp\left\{-\int_0^1 \frac{1}{\sqrt{x^2-4\delta sx-4s^2}} \times\left(\delta+\frac{2(\delta^2+1)(\eta-1)s}{x-2\delta s+\sqrt{x^2-4\delta sx-4s^2}}\right)ds\right\},$$

$$= \exp\left\{-\int_0^1 \frac{1}{\sqrt{x^2-4\delta sx-4s^2}} \times\left([(\eta-1)i+\delta]+\frac{\eta-1}{2}\frac{x-2is-\sqrt{x^2-4\delta sx-4s^2}}{s}\right)ds\right\},$$

where

$$\tilde{p}_n(x)=\sqrt{(\eta)_n/n!}\; p_n(x)=\frac{(\delta^2+1)^{n/2}}{n!} M_n(x;\delta,\eta).$$

Use

$$\int_0^1 \frac{ds}{\sqrt{x^2-2\delta sx-4s^2}} = \frac{1}{2i}\int_0^1 \frac{\frac{d}{ds}([\delta x+2s]i+\sqrt{x^2-4\delta sx-4s^2})}{[\delta x+2s]i+\sqrt{x^2-4\delta sx-4s^2}}ds = \frac{1}{2i}\log\left\{\frac{[\delta x+2]i+\sqrt{x^2-4\delta sx-4}}{(1+\delta i)x}\right\}$$

and

$$\int_0^1 \frac{x-2is-\sqrt{x^2-4\delta sx-4s^2}}{\sqrt{x^2-4\delta sx-4s^2}}\frac{ds}{s}$$

$$= 4x(\delta-i)\int_0^1 \frac{1}{x-2is+\sqrt{x^2-4\delta sx-4s^2}} \times \frac{\frac{d}{ds}(x+2is+\sqrt{x^2-4\delta sx-4s^2})}{-2\delta x-4s+2i\sqrt{x^2-4\delta sx-4s^2}}ds$$

$$= -2\int_0^1 \frac{d(x+2is+\sqrt{x^2-4\delta sx-4s^2})}{x+2is+\sqrt{x^2-4\delta sx-4s^2}}$$

$$= -2\log\left\{\frac{x+2i+\sqrt{x^2-4\delta x-4}}{2x}\right\},$$

to find the asymptotic formula

$$\lim_{n \to \infty} 2^{n+1} \sqrt{\pi n} \frac{M_n(x; \delta, \eta)}{n!} \{x - 2\delta + \sqrt{x^2 - 4\delta x - 4}\}^{-n}$$

$$\times \left\{\frac{\delta x + 2 + i\sqrt{x^2 - 4\delta x - 4}}{x(i - \delta)}\right\}^{\eta x i/2}$$

$$= \{x - 2\delta + \sqrt{x^2 - 4\delta x - 4}\}^{1/2} \{x^2 - 4\delta x - 4\}^{-1/4}$$

$$\times \left\{\frac{\delta x + 2 + i\sqrt{x^2 - 4\delta x - 4}}{x(i - \delta)}\right\}^{(n-1)/2}$$

$$\times \left\{\frac{x + 2i + \sqrt{x^2 - 4\delta x - 4}}{2x}\right\}^{n-1}$$

which holds uniformly on every compact subset of $\mathbb{C} \setminus [b - 2a, b + 2a]$. This is, as far as we know, the first time an asymptotic formula (of Plancherel–Rotach type) is given for these orthogonal polynomials.

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